Fuzzy Field Theory as Random Matrix Model

Thesis Proposal

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Contents

1 Introduction 4
  1.1 The Fuzzy Sphere ............................................. 4
  1.2 Scalar fields on the Fuzzy Sphere ......................... 6
  1.3 Random matrices and the Wigner distribution ........ 7

2 Results 9

3 Proposed research goals 11

4 Conclusion 12
Abstract

This thesis proposal summarizes previous research results and outlines the goals of further research which are going to form the linear basis of the future PhD thesis.

First, we shortly introduce the basics concepts of the problematics, namely the notion of the fuzzy sphere, scalar fields on the fuzzy sphere and random matrices. We refer to the literature for more detailed treatment.

Then, we present our current results. We have expressed the scalar field theory on the fuzzy sphere as a random matrix model and we have shown that the eigenvalue distribution of the field in the large $N$ limit remains the Wigner semicircle even after the addition of the kinetic term. We have analyzed the observables of the theory and we have obtained the joint distribution in the case involving two types of matrices.

Finally, we conclude with the proposed research. We outline several areas of research that are and will be pursued. We shortly describe how we expect to implement or generalize our ideas and what kind of findings we expect.
1 Introduction

1.1 The Fuzzy Sphere

In this section, we will briefly introduce the notion of non-commutative spaces with emphasis on the non-commutative version of the two sphere. More details and proofs are found in the literature [1].

Every manifold comes with a naturally defined associative algebra of functions with point-wise multiplication. This algebra is generated by the coordinates of the manifold and is from the definition commutative. As it turns out, this algebra contains all the information about the original manifold and we can describe geometry of the manifold purely in terms of the algebra. Also, every commutative algebra is an algebra of functions on some manifold. Therefore, what we get is

\[
\text{commutative algebras} \longleftrightarrow \text{differentiable manifolds}.
\]

A natural question to ask is whether there is a similar expression for non-commutative algebras, or

\[
\text{non-commutative algebras} \longleftrightarrow ???
\]

The obvious answer is no, there is no space to put on the other side of the expression. Coordinates on all the manifolds commute and that is the end of the story. So, as is often the case, we define new objects, called non-commutative manifolds, that are going to fit on the right hand side. Namely we look how aspects of the regular commutative manifolds are encoded into their corresponding algebras and the object, which would be encoded in the same way in a non-commutative algebra is defined as a non-commutative manifold.

This is going to introduce non-commutativity among the coordinates. This notion should not be completely new, as the reader probably recalls the commutation relations of the quantum mechanics \([x^i, p_j] = i\hbar\delta^i_j\). In classical physics, the phase space of the theory was a regular manifold. However in quantum theory we introduce non-commutativity between (some) of the coordinate and therefore the phase space of the theory becomes non-commutative. One of the most fundamental consequences of the commutation relations is the uncertainty principle. The exact position and momentum of the particle can not be measured and therefore we can not specify a single particular point of the phase space. Similarly, if there is non-commutativity between the coordinates, there is a corresponding uncertainty principle in measurement of coordinates. The notion of a space-time point does not make sense anymore, since we can not exactly say, where we are. This introduces a short distance structure to the space, quantities at short distance are not well localized. This is reflected in the name 'fuzzy spaces', which refers to non-commutative manifolds with a finite-dimensional underlying Hilbert space. \(^1\)

In practice, we often 'deform' a commutative space into its non-commutative analogue. In this way we get non-commutative spaces that give a desired commutative limit. An example of such deformation is already mentioned phase space of quantum mechanics, but to illustrate the idea better, and since we will need the notions later, let us show how this works for a two-sphere.

\(^1\)In the Encarta dictionary, one of the meanings of the word fuzzy is: blurred, not sharp enough to be seen or heard clearly.
The regular two-sphere is defined as the set of points with a given distance from the origin, i.e. \( \sum_{i=1}^{3} x_i^2 = R^2 \). This comes with an understood condition on commutativity of the coordinates \( x_i x_j - x_j x_i = 0 \). Coordinates constrained in this way generate the algebra of all the functions on the sphere.\(^2\)

Now we define the fuzzy two-sphere by the coordinates \( \hat{x}_i \), which obey the following conditions

\[
\sum_{i=1}^{3} \hat{x}_i^2 = \rho^2, \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i \theta \varepsilon_{ijk} \hat{x}_k, \tag{1.1}
\]

where \( \rho, \theta \) are parameters describing the fuzzy sphere, in a similar way as \( R \) did describe the regular sphere. The radius of the original sphere was encoded in the sum of the squares of the coordinates, so we will call \( \rho \) the 'radius' of the non-commutative sphere. We see, that such \( \hat{x} \)'s are achieved by a spin-\( j \) representation of the \( SU(2) \). If we chose

\[
\hat{x}_i = \frac{2r}{2j+1} L_i, \quad [L_i, L_j] = i \varepsilon_{ijk} L_k, \quad \sum_{i=1}^{3} L_i^2 = j(j+1) \tag{1.2}
\]

we get

\[
\sum_{i=1}^{3} \hat{x}_i^2 = \left( 1 - \frac{1}{N^2} \right) r^2, \quad \theta = \frac{2r}{N}, \tag{1.3}
\]

with \( N = 2j + 1 \) the dimension of the representation. Matrices \( \hat{x}_i \) become coordinates on the non-commutative sphere. Note, that the limit \( N \to \infty \) removes non-commutativity, since \( \theta \to 0 \), and we recover a regular sphere with radius \( r \). This explains a rather strange choice of parametrization in (1.2). Also note, that this way we got a series of spaces, one for each \( j \) (or \( N \)). The important fact is that the coordinates still do have the \( SU(2) \) symmetry and therefore it makes sense to talk about this object as spherically symmetric. This explains the particular choice of deformation in (1.2).

The non-commutative analogue of the derivative is the \( L \)-commutator, since it captures the change under a small translation, which is rotation in the case of the sphere. The integral of a function becomes a trace, since it is a scalar product on the space of matrices. It can be seen that both these have correct commutative limits.

We need to show, that basis defined by (1.2) and (1.3) is complete. It can be shown that the previous formulation is equivalent to the following one, which guarantees the completeness. Spherical harmonic functions \( Y_{lm} \) form a basis of the algebra of functions on the regular sphere. These are labeled by \( l = 0, 1, 2, \ldots \) and by \( m = -l, -l + 1, \ldots, l-1, l \) and this basis is infinite. If we truncate this algebra, namely take the following set of functions

\[
Y_{lm}^m, \quad m = -l, -l + 1, \ldots, l-1, l \quad \& \quad l = 0, 1, \ldots, N - 1 \tag{1.4}
\]

we recover a different algebra. This is obviously not the algebra of functions on the regular sphere and also to make this algebra closed, it can be seen that we need to introduce some

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\(^2\)Note that this is technically not the easiest way to do so. It is easier to introduce only two coordinates \( \theta, \varphi \) on the sphere and define the algebra of functions not by the generators, but by the basis, e.g. the spherical harmonics. However the two sphere defined in our way is easier deformed into the non-commutative analogue.
non-trivial commutation rules. So what ever it is that it has this algebra as its algebra of functions we call the fuzzy sphere.

Here we can see in a different way why the fuzzy-ness introduces short distance structure. \( l \) measures the momentum of the mode and cutting off the modes we have introduced the highest possible momentum. This in turn introduces the shortest possible distance to measure.

### 1.2 Scalar fields on the Fuzzy Sphere

As in the case of the regular sphere, the scalar field on the fuzzy sphere is a power series in the coordinate functions and thus an element of the algebra discussed in the previous section. It is itself an \( N \times N \) hermitian matrix, which can be expressed as

\[
M = \sum_{l,A} c^l_A T^l_A , \quad l = 0, 1, \ldots, N - 1 , \quad A = 1, 2, \ldots, 2l + 1
\]  

(1.5)

where \( T^l_A \) are polarization tensors, analogue of the spherical harmonic functions

\[
[L_\alpha, [L_\alpha, T^l_A]] = l(l+1)T^l_A \quad , \quad Tr \left( T^l_A T^l_B \right) = \delta^{ll'} \delta_{AB} \quad , \quad \sum_A \left( T^l_A T^l_A \right)_{ij} = \frac{2l+1}{N} \delta^{ll'} \delta_{ij} \cdot
\]  

(1.6)

The field theory is defined by the action and functional correlation functions. In what follows we work with the euclidean signature. In the analogy with the regular sphere, we define the action for the scalar field to be

\[
S_0 = \frac{1}{N} Tr \left[ -\frac{1}{2} [L_\alpha, M] [L_\alpha, M] + \frac{1}{2} \mu^2 M^2 \right] = \frac{1}{2N} Tr \left( M [L_\alpha, [L_\alpha, M]] \right) + \frac{1}{2N} \mu^2 Tr \left( M^2 \right)
\]  

(1.7)

with \( L_\alpha \) the \( SU(2) \) generators in the \( N \) dimensional representation. We will rescale the fields to absorb the \( 1/N \) factor. Using the expansion (1.5) the free field action becomes

\[
S_0 = \sum_{A,l} \frac{1}{2} \left( l(l+1) + \mu^2 \right) (c^l_A)^2
\]  

(1.8)

The correlator of the two components of the field is

\[
\langle c^l_A c^{l'}_{A'} \rangle = \frac{1}{l(l+1) + \mu^2} \delta^{ll'} \delta_{AB}
\]  

(1.9)

which is expression analogous to the usual propagator \((p^2 - m^2)^{-1}\). Using this, one can compute the free field correlation functions. The full interacting field action is then given as \( S = S_0 + S_I \), with \( S_I = \sum g_n Tr(M^n) \).

As mentioned before, the field theory is defined by the functional correlations

\[
\langle F[M] \rangle = \frac{\int dM \ e^{-S} F[M]}{\int dM \ e^{-S}} ,
\]  

(1.10)

\footnote{For example because there is an operator that takes \( Y_l \) to \( Y_{l+1} \) and to close the algebra at \( l = N - 1 \) this operator needs to vanish for such \( l \). A familiar story from quantum mechanics.}
or by the generating function for the correlators

\[ Z(J) = \frac{1}{\int dM \ e^{-S}} \int dM \ e^{-S + \text{Tr}(JM)}, \tag{1.11} \]

or in any other usual way. One then derives the Feynman rules and Feynman diagrams. There is however one important difference for the case of the non-commutative field theory, which we will just briefly mention here, with more details to be found in the references.

Vertices brought by the interaction terms will not have the full permutation symmetry among the legs. Therefore some diagrams, that gave the same contribution in the case of the regular field theory are going to be different now and need special treatment. The words to look for are double line notation, planar and non-planar graphs, UV/IR mixing.

1.3 Random matrices and the Wigner distribution

In this section, we briefly introduce the subject of random matrices. More details and rigor can be found in references listed [2].

What we will refer to as random matrices are just matrix valued random variables, or matrices with random entries. These are defined as a set of matrices and a function, that determines the probability of each element.

We will deal with the unitary ensembles, for which the random matrix \( M \) is an \( N \times N \) hermitian matrix and the probability is invariant under transformation \( M \rightarrow U^\dagger MU \) for any \( U \in SU(N) \). Together with the condition on statistical independence of linearly independent elements of \( M \), this constraints the probability density greatly to be \( P(M) = \exp(-a\text{Tr}M^2 + b\text{Tr}M + c) \) with \( a \) real and positive and \( b,c \) real. So in the most general case of hermitian random matrix ensemble, expectation value of any function of the random matrix \( M \) is given by

\[ \langle f(M) \rangle = \int dM \frac{\exp(-a\text{Tr}M^2 + b\text{Tr}M + c)}{Z} f(M), \tag{1.12} \]

where

\[ dM = \prod_i dM_{ii} \prod_{i<j} d\text{Re}(M_{ij})d\text{Im}(M_{ij}) \]

is the volume element on the space of hermitian matrices and \( Z \) is the normalization

\[ Z = \int dH \exp(-a\text{Tr}M^2 + b\text{Tr}M + c), \tag{1.13} \]

such that \( \langle 1 \rangle = 1 \).

Not surprisingly, there are numerous important results in the theory of random matrices. Here, we will state just one that is going to be important for the next section. Namely, we will compute the distribution of the eigenvalues of matrix \( M \) in the limit of very large \( N \) in the case of \( b = c = 0 \). Also, we will denote \( a = \mu^2/2 \), so in this case

\[ P(M) = \exp\left(-\frac{1}{2}\mu^2\text{Tr}M^2\right), \tag{1.14} \]
To obtain the result, we first note, that we can diagonalize the matrix $M$ by writing

$$M = U^\dagger \Lambda U,$$

where $\Lambda$ is the diagonal matrix of the eigenvalues $\lambda_i$, $i = 1, 2, \ldots, N$ of $M$. The probability does not depend on $U$ and the integration measure becomes

$$dM = dU \Delta^2(\lambda) \prod_{i=1}^{N} \lambda_i,$$

where $\Delta(\lambda)$ is the Vandermonde determinant

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j).$$

The $dU$ integration is now trivial and $\Delta(\lambda)$ can be exponentiated into the measure itself so we can turn the original matrix problem into a problem of gas of $N$ particles with the following effective action

$$S_{\text{eff}} = \frac{1}{2} \mu^2 \sum_i \lambda_i^2 - 2 \sum_{i<j} \log |\lambda_i - \lambda_j|.$$  

Careful analysis shows, that in the large-$N$ limit this expression is of order $N^2$ and therefore the path integral is dominated by the solution of the classical equation of motion $\partial S_{\text{eff}}/\partial \lambda_i = 0$. We formally define the eigenvalue density as

$$\rho(\lambda) = \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

and we expect this to become continuous in the large-$N$ limit. Also in this limit for any function $f$ we get $\sum_i f(\lambda_i) \rightarrow \int f(\lambda) \rho(\lambda) d\lambda$. Now follows a technical part where the equations of motion for the $N$ particle theory are rewritten in the terms of this eigenfunction density and solved. We will just state the the resulting expression, which is

$$\rho(\lambda) = \frac{2\mu^2}{\pi} \sqrt{\frac{N}{\mu^2} - \lambda^2} \quad \lambda \in \left( -\frac{\sqrt{N}}{\mu}, \frac{\sqrt{N}}{\mu} \right)$$

and is called the Wigner semicircle law. We see, that the density is a semicircle of the radius $\sqrt{N}/\mu$, with zero probability outside and integrates to the number of effective particles $N$. To obtain a finite result, we can rescale $M \rightarrow M/\sqrt{N}$.

The averages of the original matrix ensemble (1.14) are then given by

$$\langle \text{Tr}(M^k) \rangle = \int d\lambda \lambda^k \rho(\lambda).$$
2 Results

Comparing (1.10) and (1.12) we see, that computing averages in the random matrix theory and computing correlation functions in the fuzzy field theory with a corresponding action are essentially the same calculations. The Gaussian ensemble (1.14), which leads to the Wigner distribution, corresponds after the rescaling to the case of an infinitely massive scalar field on a non-commutative sphere. A natural question is whether it is possible to find matrix ensemble that would correspond to the case of a finitely massive or even massless field. Also, such models are going to involve Laplacian terms as kinetic part. These carry information about the geometrical structure of the underlying fuzzy space and it is natural to look for ensembles which reflect this structure. They were studied in [3] and this section presents a short summary of the result.

There is however a clear problem. As mentioned, the action that would correspond to a massive scalar field is of the form

\[ S = \frac{1}{2} Tr(M[L_\alpha,[L_\alpha,M]]) + \frac{1}{2} \mu^2 Tr(M^2), \]  

(2.1)

This action is not \( SU(N) \) invariant, therefore diagonalization (1.15) will leave the angular degrees of freedom present and the integration over \( dU \) is going to be highly nontrivial. [4] presents an attempt to perform this integration explicitly treating the kinetic term as a perturbation. This leads to a polynomial correction to the Wigner distribution. This is at odds with the presented result, suggesting the failure of the perturbative approach. The kinetic term becomes dominant in the large-\( N \) limit.

A rather different approach is used in [3]. We look for a joint distribution \( \rho(x,y) \) for the eigenvalues of the matrices \( M \) and \( B = [L_\alpha,[L_\alpha,M]] \). Rather than writing down an equation of motion for this probability as in the case the Wigner distribution, we write recursion rules for the expectation values \( \langle Tr(M^mB^b) \rangle \) and solve these.\(^4\) This is done by defining the two dimensional generating function

\[ \phi(s,t) = \sum s^m t^b W_{m,b}, \]  

(2.2)

where \( W_{m,b} \) is \( \langle Tr(M^mB^b) \rangle \) after appropriate rescaling. Then, we compute the distribution \( \rho(x,y) \) by

\[ W_{m,b} = \int dx \, dy \, x^m y^b \rho(x,y). \]  

(2.3)

If we write

\[ \langle MM \rangle_{ij} = f \delta_{ij} \quad , \quad \langle MB \rangle_{ij} = h \delta_{ij} \quad , \quad \langle BB \rangle_{ij} = g \delta_{ij}, \]  

(2.4)

the joint probability distribution for the eigenvalues of \( M \) and \( B \) is then

\[ \rho(x,y) = \rho_R(x) \rho_{R'}(y) \frac{1 - \gamma^2}{(1 - \gamma^2) - 4\gamma(1 + \gamma^2)xy + 4\gamma^2(x^2 + y^2)^2}, \]  

(2.5)

where \( \gamma = h/\sqrt{fg} \). The result is thus a product of two Wigner distributions which are correlated in a nontrivial way. The radii of the Wigner semicircles are \( R = 2\sqrt{f}, R' = 2\sqrt{g} \).

\(^4\)These can be obtained also as Schwinger-Dyson equations for the corresponding correlators.
For $\gamma^2 \leq 1$ the distribution (2.5) is always positive and the correlation of $x$ and $y$ with such distribution turns out to be $\gamma$. Indeed for $\gamma = 0$ we get two independent Wigner distributions, for $\gamma = 1$ we get $\rho(x, y) = \rho(x)\delta(x - y)$ and for $\gamma = -1$ we get $\rho(x, y) = \rho(x)\delta(x + y)$, i.e. uncorrelated, fully correlated and fully anticorrelated $x, y$.

This approach uses only the two point functions of the theory, so it can be used for a more general action. We see, that the action (2.1) can be expressed as
\[ S = \sum_{l,A} \frac{1}{2} G^{-1}(l) \left( c_A^l \right)^2, \]
where $G(l)$ is the $l$-dependent part of the propagator. So specifying the propagator of the theory is enough to determine $\gamma$. If $G(l) \sim l^\alpha$ for large $l$, we can compute $\gamma$ to vanish in the large-$N$ limit if $\alpha \in (-2, -6)$ and stay finite for other values.

The case $\alpha = 0$ corresponds to the original Gaussian ensemble. We recover the original Wigner result as we should and we also get $\gamma = \sqrt{3}/2$, so the eigenvalues of $M$ and $B$ stay correlated.

The case $\alpha = -2$ is borderline and corresponds to the massless field. For such $\alpha$ we get a very slow vanishing $\gamma$, namely $\gamma \sim 1/\sqrt{\ln N}$. However the limit $\mu \to 0$ deserves special attention, since in this case $f$ diverges. This is because the $l = 0$ mode, i.e. the trace mode of the field, has vanishing action, thus contributes to all orders in $N$ and is dominant. We need to remove it by redefining
\[ \tilde{M} = \sum_{l>0,A} c_A^l T_A^l = M - \frac{1}{\sqrt{N}} Tr M \]
and considering traceless part $\tilde{M}$ as the field. In that case we recover the Wigner distribution with $f = 2 \ln N/N$.

Authors of [5] also formulated field theory on the fuzzy sphere as a matrix model. However their treatment is different from the present one, since the theory is reduced to the regular matrix model with no Laplacian term.
3 Proposed research goals

In this section, we explain several ways how we are going to generalize the above results and how we will use these results.

- The first step is to compute expectation values and distributions for more general observables, such as

\[ \text{Tr} \left( M^a B^b M^c B^d \right), \text{Tr} \left( M^a[L_\alpha, M]^b M^c[L_\alpha, M]^d \right), \ldots \]  

(3.1)

Obviously the choice of the observables is going to be motivated by the desired physical applications discussed below.

Also we will investigate observables involving more general functions of the random matrix \( M \).

- Another way to generalize the results is to consider different underlying spaces than the (fuzzy) sphere. Choosing a different set of matrices \( L_\alpha \) in the action for our field (1.7) will yield field theory on a different space, that corresponds to the chosen set of matrices. The fuzzy torus and fuzzy projective spaces are the obvious candidates.

With the field theory on such spaces being different from the theory on the fuzzy sphere we expect this difference to be reflected in the results.

- The generalization of the free field theory into an interacting one is discussed in the introductory sections and is usually treated perturbatively. Generalizing our results into the interacting realm could provide an alternative, and essentially non-perturbative, approach.

- If we look for the probability distribution of vacuum fluctuations of an observable \( A \), we can define such quantity in the following way

\[ P(\alpha) d\alpha = d\alpha \langle 0 | \delta(\alpha - A) | 0 \rangle \]  

(3.2)

It is straightforward to check that such probability distribution gives the expected moments \( \int \alpha^n P(\alpha) d\alpha \). Expanding the delta function we get

\[ P(\alpha) = \frac{1}{2\pi} \int d\lambda e^{i\alpha \lambda} \sum_n \frac{(-i\lambda)^n}{n!} \langle A^n \rangle \]  

(3.3)

This treatment is general for any field theory. We can now consider a field theory on the fuzzy sphere (or a more general space) and using this formula compute vacuum fluctuation distributions for different observables of the theory. Powers of the field \( M^m \) and the energy-momentum tensor are the prime examples of physically interesting observables.

We will also interpret fuzzy space description as a starting point for regularization of the vacuum expectation values for the continuous theories and we expect the fluctuations of the energy-momentum tensor to be relevant for quantum gravity questions. There are previous results considering vacuum fluctuations to compare with [6].
• Last, but not least, our results will be relevant also from purely mathematical point of view. New matrix ensembles in random matrix theory and correlated random variables in free probability theory are the expected examples. We will look also into this aspect of our research.

4 Conclusion

In this thesis proposal, we have summarized our previous results in the scalar field theory on the fuzzy field. We have expressed this theory as a random matrix model. We have analyzed the observables of the theory and we have shown that the eigenvalue distribution of the field in the large $N$ limit remains the Wigner semicircle even after the addition of the kinetic term. We have analyzed the expectation values of mixed products of the random matrix and its Laplacian obtaining a joint probability distribution of the eigenvalues. The distribution corresponds to a two dimensional correlated Wigner distribution.

We have also presented several ways how we will generalize and implement these results. We will generalize the approach for more general observables, for different underlying spaces and for the interacting theory and look for applications in computing vacuum fluctuations, in string theory, quantum gravity and pure mathematics.
References

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