Anomalies and anomalous symmetries

Juraj Tekel juraj(dot)tekel(at)gmail(dot)com

Introductory notes about anomalies and anomalous symmetries in the quantum field theory.

December 2010

Contents

1 Introduction 1

2 Reminders - symmetries, quantum corrections, chirality 2
  2.1 Symmetries ......................................................... 2
  2.2 Quantum corrections ........................................... 3
  2.3 Chiral fields ...................................................... 4

3 The chiral anomaly 5
  3.1 Anomaly - the first meeting ..................................... 5
  3.2 The meaning of it all ............................................ 8
  3.3 The full chiral anomaly ......................................... 9

4 General treatment of anomalies 10
  4.1 Gauge anomalies .................................................. 10
  4.2 Global anomalies ................................................ 10

5 The anomaly cancellation in the Standard Model 10

6 The $\pi^0 \to \gamma\gamma$ problem 10

7 Anomalies in the functional integral language 10

8 Some notes beyond these notes 10

9 References 11

1 Introduction

In quantum field theory, we usually build on some underlying classical theory which we quantize. However the symmetries, that were present in the classical theory need not to be present in its quantized version. When quantizing, we need to introduce a regularization scheme, that handles infinities in the loop integrals. If there is no regularization that is compatible with the symmetry of the classical theory, the symmetry is not present in the quantum theory. Equivalently one can see this as the symmetry being broken by the quantum corrections to the classical theory.

Such symmetries are called anomalous. The currents, that were conserved by the classical symmetry get a non-zero divergence which is called anomaly.

This might and might not be a problem. We simply may live with a quantum theory with a smaller symmetry group. However if the symmetry was essential also in the quantum theory, we have
a problem and in general the quantum theory is no good. One needs to carefully make sure that the anomaly is not present. This is the case of gauge symmetry, which is necessary to remove the unphysical degrees of freedom and make the theory unitary.

2 Reminders - symmetries, quantum corrections, chirality

As mentioned in the previous section, anomalies arise due to the symmetries being broken by quantum corrections. So let us first remind ourselves, what we mean by these two things. Also, as we will see in the following section, chiral symmetries are going to be crucial in the discussion of anomalies, so we present a short overview of this topic also.

2.1 Symmetries

A symmetry of a classical field theory governed by the action $S = \int d^4x L(\phi, \partial\phi)$ is a transformation of the fields, that leaves the results theory intact. This means, that for an infinitesimal transformation of fields given by

$$\phi(x) \rightarrow \phi(x) + \varepsilon \delta\phi(x)$$

the Lagrangian density does not change $\delta L = 0$. Obviously, this condition on the transformation of Lagrangian and action could be relaxed quite a lot, but for our purposes this is all we need. \(^1\) At the classical level the most important consequence of any symmetry is a conservation law. We see that

$$\delta L = \frac{\partial L}{\partial \delta\phi} \delta\phi + \frac{\partial L}{\partial (\partial_{\mu}\phi)} \partial_{\mu} \delta\phi = \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu}\phi)} \delta\phi \right),$$

where we have used the equations of motion. Therefore if we denote $j_{\mu} = \frac{\partial L}{\partial (\partial_{\mu}\phi)} \delta\phi$ we see, that this current is conserved

$$\partial_{\mu} j_{\mu} = 0.$$ \(^2\)

We define charge corresponding to this current as $Q = \int d^3x j^0$ and then $\partial_t Q = 0$. Generalization to more than one field is straightforward. If the symmetry has more than one direction $\phi \rightarrow \phi + \varepsilon_i \delta_i \phi$ then we get a conserved current for every direction of the symmetry.

The charges of the symmetry then obey the algebra of generators of the symmetry group. Therefore if we identify conserved currents and conserved charges, we can reconstruct the symmetry of the theory.

In the quantized theory, conserved charges of the symmetry, when replaced by corresponding operators, provide a natural representation of the symmetry group on the Hilbert space of the quantum theory.

If the reader is not familiar with some examples, we strongly suggest to go over couple of exercises that are available in the referred literature.

\(^1\)For example we could have $\delta L = \partial_{\mu} K^\mu$ for some $K^\mu$. Then this change would yield a surface term in the action and this extra term would vanish for $K$ falling of fast enough. The action does not need to remain the same, since the equations of motion are given by the extremum of the action and not the action itself.
2.2 Quantum corrections

Classical and quantum theories obviously do differ. If we consider the classical theory as a cornerstone that the quantum theory is build on, then this difference between the theories is usually referred to as quantum corrections. The classical theory is governed by the action $S$ and the quantum theory, being different, is given by corrected action $\Gamma$. This is called the quantum effective action.

In quantum theory, the Feynman diagrams are generated by the generating functional $Z[J]$. This however generates all the diagrams with vacuum bubbles and disconnected diagrams. It can be show, that if we define $W[J] = -i \log Z[J]$, this generates only the diagrams which are connected.

Now, we perform a legendre transform on $W$ we get

$$\Gamma[\phi] = W[J] - \int d^4x J(x) \phi(x),$$

where $\phi = \delta W/\delta J, J = \delta \Gamma/\delta \phi$. After a careful check, one sees that $\Gamma$ generates one particle irreducible vertexes

$$\Gamma[\phi] = \sum_{N} \frac{1}{N!} \int d^4x_1 \ldots d^4x_N \phi(x_1) \ldots \phi(x_N) V_{1\ldots N}(x_1, \ldots x_N).$$

To see, that $\Gamma$ is indeed an action of the quantum theory, we express the $S$-matrix functional, which defines the theory, in terms of $\Gamma$. This turns out to be

$$S = e^{i\Gamma[\phi]} \bigg|_{\delta \Gamma/\delta \phi = 0}.$$  

This means, that we solve for fields, that extremize the quantity $\Gamma$ and then let thing evolve with expression $e^{i\Gamma}$ where this solutions are inserted. Could not work more like an action.

To see, that $\Gamma$ indeed reflects quantum corrections, we reinsert factors of $\hbar$ into the generating functional. After some graph-theory we see that

$$\Gamma[\phi] = \sum_{L=0}^{\infty} \hbar^L \Gamma^{(L)}[\phi],$$

where $L$ is number of the loops in the diagram and $\Gamma^{(L)}$ is the part of the effective action corresponding to the diagrams with $L$ loops. Further more, it can be shown that $\Gamma^{(0)} = S$, i.e. the tree level diagram reproduce the classical theory and $L$-loop diagrams give $L$-th order quantum corrections to the theory.

To make this discussion little more clear, let us consider a very simple theory, $\phi^4$ theory given by the classical action

$$S = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4.$$  

The 0-th order (in $\hbar$) contribution to $\Gamma$ is given by this expression. 1-st order contribution to $\Gamma$ is given by all the one loop diagrams. For propagation of a single particle, this is the following bubble diagram.
Therefor, we get for $\Gamma^{(1)}$

$$\Gamma^{(1)}[\phi] = \int d^4x \phi^2(x)(-i\lambda) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2}. \quad (2.9)$$

Putting $\Gamma^{(0)}$ and $\Gamma^{(1)}$ together, we can see that this correspond to a first order quantum correction to the mass of the particles. Similarly, for the candy diagram we would get a quantum correction to the coupling constant and also corrections to the amplitudes of the processes involving more $\phi$-particles.

The curious reader can find much more and precise details in the book by V.P. Nair.

2.3 Chiral fields

Recall the free Dirac fermions are governed by the following Lagrangian

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi, \quad (2.10)$$

from which the usual Dirac equation follows

$$(i\gamma^\mu \partial_\mu - m) \psi = 0. \quad (2.11)$$

In a suitable representation of $\gamma$ matrices it is explicit that the spinor $\psi$ can be written in terms of two parts $(\psi_1, \psi_2)$ such that the Dirac equation becomes

$$\begin{pmatrix} -m & i(\partial_0 + \sigma \cdot \nabla) \\ i(\partial_0 + \sigma \cdot \nabla) & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad (2.12)$$

where $\sigma$ are the Pauli matrices. One could see this at the level of the Lagrangian independently of the representation of $\gamma$ matrices we use. If we define

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi, \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi \quad (2.13)$$

we see that the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R + m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L). \quad (2.14)$$

If we deal with massless fermions, $m = 0$, the left and right components of the spinor decouple and live their own lives. These components are referred to as left and right chiral components of the spinor $\psi$. Note that left and right components of a spinor live in completely different representations of Lorentz group.

One can check, that $\psi \rightarrow e^{i\alpha} \psi$ is a symmetry of the theory. Current corresponding to this symmetry is given by $j^\mu = \bar{\psi} \gamma^\mu \psi$ and is referred to as vector current. For massless theory, also $\psi \rightarrow e^{i\alpha \gamma^5} \psi$ is a symmetry and the corresponding axial current is given by $j^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$. This latter symmetry is what is usually called the chiral symmetry.

A very nice discussion of the chiral fields is given in lecture notes by David Tong.

$^2\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
3 The chiral anomaly

One might ask why do we chose such a peculiar example to start with in an introductory text. The reason is following. As mentioned in the introduction, symmetry is going to be anomalous if there is no regularization procedure, that respects the symmetry. However, for most of the theories dimensional regularization or Pauli-Villars regularization provide such regularization. And chiral symmetry being the exception. The dimensional regularization does not work, since there is no analogue of $\gamma^5$ in $4 + \varepsilon$ dimensional space and thus the chiral symmetry is not well defined. The Pauli-Villars regularization effectively introduces a mass term, which breaks the chiral symmetry. The lattice regularization seems to provide a tool that obeys all the symmetries, however chiral symmetries do find a subtlety to avoid also this procedure. Later in the text we will learn how to see that the anomaly is real and there is truly no regularization that removes it.

3.1 Anomaly - the first meeting

After a lot of introduction we are finally ready to see the anomaly for the first time.

Let us have a free massless chiral left handed fermions. This theory is given by a simple Lagrangian

$$L = \bar{\psi}_L \gamma^\mu \partial_\mu \psi_L.$$  \hfill (3.1)

We introduce an interaction term $-e \bar{\psi}_L \gamma^\mu A_\mu \psi_L$. We will discuss the meaning of this term later, for now lets study the consequences.\(^3\) The new Lagrangian is

$$L = \bar{\psi}_L i \gamma^\mu (\partial_\mu + i e A_\mu) \psi_L$$  \hfill (3.2)

and the extra term corresponds to a $\bar{\psi}\psi A$ vertex of the following factor

$$-ie\gamma^\mu \frac{1}{2} (1 - \gamma^5)$$  \hfill (3.3)

Here, we have introduce a projector on to the left chirality to make sure only this component of the fields interacts.

This new term gives a $A \to AA$ process, given by the following amplitude

\[ \crossedDiagram \]

\[ (k_2, \nu) \leftrightarrow (k_3, \lambda) \]

\(^3\)The fields $A_\mu$ would correspond to the gauge fields if we wished to make the global symmetry $\phi \to e^{i\alpha} \phi$ local. However to make this procedure complete, would have to introduce a kinetic term for the gauge fields. We will discuss this possibility later.
\[-i\mathcal{M}_{\mu\nu\lambda} = -ie^3 \int \frac{d^4p}{(2\pi)^4} \left\{ Tr \left[ \gamma_\mu \frac{i}{p + k_2/2} \gamma_\nu \gamma_\lambda \frac{i}{p - k_3/2} (1 - \gamma^5) \right] + (k_2, \nu) \leftrightarrow (k_3, \lambda) \right\} \quad (3.4)\]

We have combined all the projections on the left-handed fermions into one.

The field \( A_\mu \) has spin 1 and thus we need the amplitude \( \mathcal{M} \) to be symmetric under permutation of the incoming particles, yielding the Bose statistics for these particles.

It is crucial to note, that if we want to promote the field \( A_\mu \) to a gauge field and a dynamical variable, we need to ensure the gauge invariance of the theory. \( A_\mu \) would then be the product of gauging the obvious \( \psi_L \rightarrow e^{i\alpha} \psi_L \) symmetry of the original Lagrangian. One way to get this, is to require the Ward-Takashi identities to hold, namely to require

\[ k_1^\mu \mathcal{M}_{\mu\nu\lambda} = k_2^\nu \mathcal{M}_{\mu\nu\lambda} = k_3^\lambda \mathcal{M}_{\mu\nu\lambda} = 0. \quad (3.5) \]

**Excercise 1.** Show, that amplitude (3.4) is invariant under any exchange of the incoming particles and it obeys the Ward-Takahashi identity. For the latter, you will need to use the identity \( k_1 + k_2 + k_3 = 0 \).

Note the shift of integration variables you had to perform in order to obtain this.

As you can expect now, the shift of momenta in the integration is going to be the source of trouble and we can not have both the Bose statistics and gauge invariance. Let’s see this explicitly.

To impose the Bose statistics, we will consider another two pairs of diagrams so that the over amplitude is sum over all permutations of external photos.

**Excercise 2.** Show, that this is equivalent to changing the loop momentum routing as pictured

Also show, that the resulting amplitude is the following expression.
We compute the amplitude to be

\[-i\mathcal{M}_{\mu\nu\lambda} = -\frac{e^3}{3} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma^\mu \gamma^\nu \gamma^\lambda \left( \frac{1}{p + k_2} - \frac{1}{p + k_1} \right) \gamma_4 \right] \]

\[+ \gamma^\mu \gamma^\nu \gamma^\lambda \left( \frac{1}{p + k_2} - \frac{1}{p + k_1} \right) \gamma_4 \left( \frac{1}{p + k_3} - \frac{1}{p + k_4} \right) \gamma_5 \]

Using Furry’s theorem, we can drop terms that do not involve \( \gamma_5 \) in this formula.\(^\text{4}\) We dot \( k_1^\mu \) into this expression to obtain

\[-i k_1^\mu \mathcal{M}_{\mu\nu\lambda} = -\frac{1}{6} e^3 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \left( \frac{1}{p + k_2} - \frac{1}{p + k_1} \right) \gamma^\nu \gamma^\lambda - \frac{1}{p + k_3} \gamma^\nu \frac{1}{p + k_4} \gamma^\lambda \right] \]

\[+ \frac{1}{p + k_2} \gamma^\nu \frac{1}{p + k_3} \gamma^\lambda - \frac{1}{p + k_4} \gamma^\nu \frac{1}{p + k_4} \gamma^\lambda \right] \gamma_5 \right]. \quad (3.6)\]

Here, we have written \( k_1 = p - k_2 - (p + k_3) \).

Now note, that if we shift \( p \rightarrow p + k_2 - k_1 \) and \( p \rightarrow p + k_3 - k_1 \) in the second terms of first and second row respectively, we get cancellation between the terms in each row. However there is a catch in this.

These integrals are divergent, so shifting of the integration variable might bring up a surface term. To see this, we compute for a general \( f \)

\[\int \frac{d^4p}{(2\pi)^4} f(p^\mu + a_\mu) = \int \frac{d^4p}{(2\pi)^4} f(p^\mu) + a_\mu \int \frac{d^4p}{(2\pi)^4} \partial_\mu f(p^\mu) + \ldots . \quad (3.8)\]

The higher order terms do not contribute for functions falling of fast enough and later due to the gauss theorem.

We see, that if \( f \) does not fall off fast enough, the second term is nonzero. We compute the extra term.

\[a_\mu \int \frac{d^4p}{(2\pi)^4} \partial_\mu f(p^\mu) = -i \int \frac{d^4p}{(2\pi)^4} a_\mu \frac{\partial}{p^E} f(p^E) =
\]

\[= -i a_\mu E \int_0^\Lambda \int p^2 d^3p E \frac{\partial}{p^E} f(p^E) =
\]

\[= -i a_\mu E \int \int \frac{d^3p}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{\partial}{p^E} f(p^E) \bigg|_{|p^E|=\Lambda} =
\]

\[= -i a_\mu E \frac{1}{8\pi^2} \lim_{|p^E| \to \infty} \langle p^2 p^E f(p^E) \rangle =
\]

\[= -i a_\mu E \frac{1}{8\pi^2} \lim_{p \to \infty} \langle p^2 p^E f(p) \rangle . \quad (3.9)\]

We have first shifted to euclidean signature and then used the gauss theorem to pick up the surface integral. We then wrote the integral as a mean value over the large 3-sphere, which becomes

\(^\text{4}\)Footnote about the Furry’s theorem.
mean value over the Lorentz group after we shift back to Minkowsky signature. Note that to do this calculation, we had to introduce a cut-off and then remove it. It is exactly the lack of symmetry of the function $f$ that makes the mean value non-zero and eventually leads to an anomaly.

Using this in (3.7) we obtain

$$-i k_1^{\mu} M_{\mu\nu\lambda} = -i \frac{e^3}{6} \int \frac{d^4 p}{(2\pi)^4} \left\{ (k_2^\nu - k_1^\nu) \frac{\partial}{\partial p^\nu} Tr \left[ \frac{1}{q} \gamma_\nu \gamma_\lambda \gamma_5 \right] + (k_3^\nu - k_1^\nu) \frac{\partial}{\partial p^\nu} Tr \left[ \frac{1}{q} \gamma_\nu \gamma_\lambda \gamma_5 \right] \right\}.$$  \(3.10\)

If there was some way how to regularize these integrals and preserve the symmetry, this expression would vanish. However since there is not, we pick up a non-zero contribution.

To simplify the formula (3.10) we will use the previous result, the fact that $\langle p_\mu p_\nu \rangle = \frac{1}{4} \eta_{\mu\nu} p^2$, that $Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_5) = 4i\epsilon_{\mu\nu\rho\sigma}$ and that $1/p = p/p^2$.

**Excercise 3.** Show, that (3.10) reduces to the following expression. Also carry out the same computation to compute $k_2^{\mu} M_{\mu\nu\lambda}$ and $k_3^{\mu} M_{\mu\nu\lambda}$.

After all the algebra, we get

$$-i k_1^{\mu} M_{\mu\nu\lambda} = \frac{e^3}{12\pi^2} \epsilon_{\nu\lambda\alpha\beta} k_2^\alpha k_3^\beta.$$  \(3.11\)

It can be proved, that this is actually complete formula, which does not get any further contributions from higher orders of perturbation theory. This result was proved by Adler and Bardeen.

**Excercise 4.** Convince yourself, preferably by means of explicit calculation, that for right handed chiral spinners we would get a minus sign in the previous formula.

### 3.2 The meaning of it all

The result of previous section deserves couple of comments.

First of all we still did not interpret the fields $A_\mu$ in any way. At the classical level, these were fields that we had to introduce in order to make the obvious symmetry $\psi_L \rightarrow e^{i\alpha} \psi_L$ of the free action local. To make them dynamical, we introduce the term $F_{\mu\nu} F^{\mu\nu}$, which is also local invariant. As expected, when going to the quantum theory, this action gets changed by the quantum corrections.

However, as one might expect from the fact that Ward-Takashi identities are broken and as we will shortly show explicitly, this corrected action is no longer invariant under the local symmetry and thus the whole quantum theory breaks down.

Therefore, $A_\mu$’s can not be treated as dynamical quantities. We could treat the fields as an external probe, a prescribed field without dynamics and life of its own. What good would such field do? We could define the current $j^{\mu}$ corresponding to the symmetry $\psi_L \rightarrow e^{i\alpha} \psi_L$ as

$$j^{\mu} = -\frac{1}{e \partial A_\mu}.$$  \(3.12\)

You can check, that for the classical theory this in fact yields $j^{\mu} = \bar{\psi}_L \gamma^{\mu} \psi_L$. However in quantum theory the action gets corrected and the current is different, with a non-vanishing divergence

$$\partial_\nu j^{\mu} = -\frac{e^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}.$$  \(3.13\)

The contribution we have computed corresponds to the term in the effective action of the following form

$$\Delta \Gamma = - \int d^4 x d^4 y \frac{e^3}{96\pi^2} \partial_{\nu} A^{\nu}(x) (\partial_{\mu} \partial^{\mu})^{-1}(x-y) \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}(y) F_{\gamma\delta}(y).$$  \(3.14\)
Where \((\partial_\mu \partial^\mu)^{-1}\) is the Green’s function of the corresponding operator. This gives a AAA vertex of factor

\[
\frac{e^3}{12\pi^2} \left( \frac{1}{k_1^2} \epsilon_{\nu \lambda \alpha \beta} k_2^\alpha k_3^\beta + (1, 2, 3) \rightarrow (2, 3, 1) + (1, 2, 3) \rightarrow (3, 1, 2) \right)
\]  

(3.15)

and this dotted with \(k_1\) gives the anomaly (3.11). This expression for the effective action yields the anomalous divergence (3.13).

Now comes a crucial observation. The extra term in the effective action is not local, it can not be expressed as one spacial integral \(\int dx\). This means, that since changing short distance behavior of the theory introduces local terms into the effective action, we can not remove this anomaly by altering the theory for short distances (e.g. introducing a very heavy particle). This shows, that there is no regularization, that does not yield an anomaly in the end, since regularization is a short distance change of the theory.

At the end, let us note that even though the name of the section might have advertised differently, what we have computed is not what is referred to as chiral anomaly in standard literature. The results we have computed are valid and will be useful, what we usually mean as chiral anomaly is the non-conservation of a current corresponding to the chiral transformation \(\psi \rightarrow e^{i\alpha \gamma^5} \psi\).

So the reader is welcomed to take the previous parts as an appetizer that was supposed to be only preparation for the main course to come.

### 3.3 The full chiral anomaly

As mentioned in the previous section, left and right chiral components yield an anomaly that differs by a minus sign. If we now define vector and axial vector currents

\[
\begin{align*}
  j^\mu_V &= j^\mu_R + j^\mu_L \\
  j^\mu_A &= j^\mu_R - j^\mu_L
\end{align*}
\]  

(3.16)

these correspond to the conserved currents of the full classical theory with both left and right handed fermions with symmetries \(\phi \rightarrow e^{i\alpha} \phi\) and \(\phi \rightarrow e^{i\alpha \gamma^5} \phi\) respectively. Now it is clear, that in the theory, where left and right components couple to the would-be gauge fields in the same way the vector symmetry is anomaly free and survives the quantization. So the vector symmetry can be consistently gauged. However the chiral symmetry is anomalous and the axial current picks up a nonzero divergence.

Here, we have studied a rather simple case of an abelian symmetry. As we will see in the next section, for a non-abelian symmetry group, the situation is going to be more complicated. However, by a clever trick, the anomaly of the vectorial symmetry can be eliminated and thus even he non-abelian symmetry can be gauged.
4 General treatment of anomalies

4.1 Gauge anomalies

4.2 Global anomalies

5 The anomaly cancellation in the Standard Model

6 The $\pi^0 \rightarrow \gamma\gamma$ problem

7 Anomalies in the functional integral language

8 Some notes beyond these notes
9 References

The number of resources one can study for the field theory is infinite. These are some texts and books that I personally like and have used in writing this text.

A great text for very introduction into the QFT are lecture notes by Martin Mojžiš. David Thing has also very good lecture notes that are usefully for the first encounter with the field theory. A book would be Peskin and Schroeder, Addison-Wesley, 1995. A great set of problems is given by Radovanović, Springer, 2008.

This text was largely inspired and guided by lecture notes on the introduction to the standard model by Daniel Kabat.