Matrix models of fuzzy field theories

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Introduction and outline

The ABEGHHK’tH mechanism

This costs too much energy! I think I'll hang out down there.
Introduction and outline

Real scalar $\phi^4$ field on plane

This costs too much energy! I think I’ll hang out down there.
In this talk, we will

- very briefly introduce fuzzy spaces and some aspects of fuzzy field theories,
- describe these theories in terms of a random matrix model,
- investigate the properties of this model.
Take home message.

- Symmetry breaking in noncommutative field theory is (very) different than in the commutative case.
- Matrix models are a great tool to analyze the(se) properties of scalar field theories on fuzzy spaces, and beyond.
Fuzzy spaces
Fuzzy sphere

- Noncommutative spaces introduce a shortest possible distance.

- Fuzzy spaces (= a finite dimensional algebra) have finite number of the "Planck cells" $N$.

- The hallmark example is the fuzzy sphere $S^2_F$.
  Hoppe '82; Madore '92; Grosse, Klimcik, Presnajder '90s

- However there are no sharp boundaries between the pieces and everything is blurred, or fuzzy.

Image from http://principles.ou.edu/mag/earth.html
Fuzzy sphere

Technically, this is done by

- truncating the possible values of $l$ in the expansion

$$f = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \phi),$$

- deforming the coordinate(s)

$$x_i x_i = \rho^2, \quad x_i x_j - x_j x_i = i\theta \varepsilon_{ijk} x_k.$$

Real functions on the fuzzy sphere are $N \times N$ hermitian and the eigenvalues of $M$ represent the values of the function on the cells.
Fuzzy scalar field theory
Scalar field theory

- Commutative euclidean theory of a real scalar field is given by an action

\[ S(\Phi) = \int dx \left[ \frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right] \]

and path integral correlation functions

\[ \langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}}. \]

- We construct the noncommutative theory as an analogue with
  - field → matrix,
  - functional integral → matrix integral,
  - spacetime integral → trace,
  - derivative → $L_i$ commutator.
Scalar field theory

**Commutative**

\[ S(\Phi) = \int dx \left[ \frac{1}{2} \Phi \Delta \Phi + \frac{1}{2} m^2 \Phi^2 + V(\Phi) \right] \]

\[ \langle F \rangle = \frac{\int d\Phi \, F(\Phi) e^{-S(\Phi)}}{\int d\Phi \, e^{-S(\Phi)}}. \]

**Noncommutative (for \( S_F^2 \))**

\[ S(M) = \frac{4\pi R^2}{N} \text{Tr} \left[ \frac{1}{2} M \frac{1}{R^2} [L_i, [L_i, M]] + \frac{1}{2} m^2 M^2 + V(M) \right] \]

\[ \langle F \rangle = \frac{\int dM \, F(M) e^{-S(M)}}{\int dM \, e^{-S(M)}}. \]

Balachandran, Kürkçüoğlu, Vaidya ’05; Szabo ’03
Spontaneous symmetry breaking
Symmetry breaking in NC field theories

- From now on $\phi^4$ theory.
- The commutative field theory has two phases in the phase diagram, disorder and uniform order phases.
  
  Glimm, Jaffe, Spencer ’75; Chang ’76
  Loinaz, Willey ’98; Schaich, Loinaz ’09

- In disorder phase the field oscillates around the value $\phi = 0$.
- In uniform order phase the field oscillates around a nonzero value which is a minimum of the potential.
Symmetry breaking in NC field theories

- The phase diagram of noncommutative field theories has one more phase. It is a non-uniform order phase, or a striped phase.
  Gubser, Sondhi ’01; G.-H. Chen and Y.-S. Wu ’02
- In this phase, the field does not oscillate around one given value in the whole space. Translational symmetry is broken.
- This has been established in numerous numerical works for variety different spaces.
  Martin ’04; García Flores, Martin, O’Connor ’06, ’09; Panero ’06, ’07; Ydri ’14; Bietenholz, F. Hofheinz, Mejía-Díaz, Panero ’14; Mejía-Díaz, Bietenholz, Panero ’14; Medina, Bietenholz, D. O’Connor ’08; Bietenholz, Hofheinz, Nishimura ’04; Lizzi, Spisso ’12; Ydri, Ramda, Rouag ’16
  Panero ’15
Symmetry breaking in NC field theories

Mejía-Díaz, Bietenholz, Panero ’14 for $\mathbb{R}^2_0$

\[ N = 35, \quad N^2 \lambda = 240, \quad N^{3/2} m^2 = -173 \]
Symmetry breaking in NC field theories

- This phase is a result of the nonlocality of the theory.
- This phase survives the commutative limit of the noncommutative theory! Result of the UV/IR mixing.
- The commutative limit of such noncommutative theory is (even more) different than the commutative theory we started with.
Symmetry breaking in NC field theories

O’Connor, Kováčik ’18 for $S_F^2$
Matrix model description of fuzzy field theories
Matrix models

- Ensemble of hermitian $N \times N$ matrices with a probability measure $S(M)$ and expectation values

$$\langle F \rangle = \frac{\int dM \, F(M)e^{-S(M)}}{\int dM \, e^{-S(M)}}.$$ 

- This is the very same expression as for the real scalar field.

- Fuzzy field theory = matrix model with

$$S(M) = \frac{1}{2} \text{Tr} \left( M [L_i, [L_i, M]] \right) + \frac{1}{2} r \text{Tr} \left( M^2 \right) + g \text{Tr} \left( M^4 \right)$$

(minus the red Brezin, Itzykson, Parisi, Zuber ’78)
Matrix models of fuzzy field theories

- The large $N$ limit of the model **without** the kinetic term

$$S(M) = \frac{1}{2} \text{Tr} (M[L_i, [L_i, M]]) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4)$$

is **well** understood.

- The key is diagonalization and the saddle point approximation.
Matrix models of fuzzy field theories

- The large $N$ limit of the model **without** the kinetic term

\[
S(M) = \frac{1}{2} \text{Tr} (M[L_i, [L_i, M]]) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4)
\]

is **well** understood.

- The key results is that for $r < -4\sqrt{g}$ we get two cut eigenvalue density.
Matrix models of fuzzy field theories

- The model with the kinetic term

\[ S(M) = \frac{1}{2} \text{Tr} (M[L_i, [L_i, M]]) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4) \]

is not well understood.

Steinacker ’05; JT Acta Physica Slovaca ’15

- The key issue being that diagonalization no longer straightforward.
The model with the kinetic term

\[ S(M) = \frac{1}{2} \text{Tr} (M[L_i, [L_i, M]]) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4) \]

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The key issue being that diagonalization no longer straightforward.

We are to compute integrals like

\[
\langle F \rangle \sim \int \left( \prod_{i=1}^{N} d\lambda_i \right) F(\lambda_i) e^{-N^2 \left[ \frac{1}{2} r \frac{1}{N} \sum \lambda_i^2 + g \frac{1}{N} \sum \lambda_i^4 - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| \right]} \\
\times \int dU e^{-N^2 \frac{1}{2} \text{Tr} (U \Lambda U^\dagger [L_i, [L_i, U \Lambda U^\dagger]])}
\]
Matrix models of fuzzy field theories

- The model with the kinetic term

\[ S(M) = \frac{1}{2} \text{Tr} (M[L_i, [L_i, M]]) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4) \]

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- The key issue being that diagonalization no longer straightforward.

- We are to compute integrals like

\[ \langle F \rangle \sim \int \left( \prod_{i=1}^{N} d\lambda_i \right) F(\lambda_i) e^{-N^2 [S_{\text{eff}}(\lambda_i) + \frac{1}{2} r \frac{1}{N} \sum \lambda_i^2 + g \frac{1}{N} \sum \lambda_i^4 - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j|]} \]

\[ e^{-N^2 S_{\text{eff}}(\lambda_i)} = \int dU e^{-N^2 \frac{1}{2} \text{Tr} (U\Lambda U^\dagger[L_i, [L_i, U\Lambda U^\dagger]])} \]

- How to compute \( S_{\text{eff}} \)?
Matrix models of fuzzy field theories

- Perturbative calculation of the integral show that the $S_{\text{eff}}$ contains products of traces of $M$. O’Connor, Sämann ’07; Sämann ’10

$$e^{-N^2S_{\text{eff}}(\lambda_i)} = \int dU e^{-N^2\epsilon \frac{1}{2}\text{Tr}(U\Lambda U^\dagger[L_i,[L_i,U\Lambda U^\dagger]])}$$

- The most recent result is Sämann ’15

$$S_{\text{eff}}(M) = \frac{1}{2} \left[ \epsilon \frac{1}{2} (c_2 - c_1^2) - \epsilon^2 \frac{1}{24} (c_2 - c_1^2)^2 + \epsilon^4 \frac{1}{2880} (c_2 - c_1^2)^4 \right] -$$

$$- \epsilon^4 \frac{1}{3456} \left[ (c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4) - 2 (c_2 - c_1^2)^2 \right]^2 -$$

$$- \epsilon^3 \frac{1}{432} \left[ c_3 - 3c_1c_2 + 2c_1^3 \right]^2$$

where

$$c_n = \frac{1}{N} \text{Tr} (M^n)$$

- The standard treatment of such multitrace matrix model yields a very unpleasant behaviour. Self interaction is way too strong in the important region.
Hermitian matrix model of fuzzy field theories

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. Steinacker '05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos '13

$$S_{\text{eff}} = \frac{1}{2} F(c_2) + \mathcal{R} = \frac{1}{2} \log \left( \frac{c_2}{1 - e^{-c_2}} \right) + \mathcal{R}$$

- Recall the perturbative action

$$S_{\text{eff}}(M) = \frac{1}{2} \left[ \varepsilon \frac{1}{2} (c_2 - c_1^2) - \varepsilon^2 \frac{1}{24} (c_2 - c_1^2)^2 + \varepsilon^4 \frac{1}{2880} (c_2 - c_1^2)^4 \right] -$$

$$- \varepsilon^4 \frac{1}{3456} \left[ (c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4) - 2 (c_2 - c_1^2)^2 \right]^2 -$$

$$- \varepsilon^3 \frac{1}{432} \left[ c_3 - 3c_1c_2 + 2c_1^3 \right]^2$$

The first line is the first terms of the small $c_2$ expansion with $c_2 \to c_2 - c_1^2$. 
Hermitian matrix model of fuzzy field theories

- For the free theory $g = 0$ the kinetic term just rescales the eigenvalues. Steinacker ’05
- There is a unique parameter independent effective action that reconstructs this rescaling. Polychronakos ’13

\[ S_{\text{eff}} = \frac{1}{2} F(c_2) + \mathcal{R} = \frac{1}{2} \log \left( \frac{c_2}{1 - e^{-c_2}} \right) + \mathcal{R} \]

- Introducing the asymmetry $c_2 \rightarrow c_2 - c_1^2$ we obtain a matrix model

\[ S(M) = \frac{1}{2} F(c_2 - c_1^2) + \frac{1}{2} r \text{Tr} (M^2) + g \text{Tr} (M^4) , \quad F(t) = \log \left( \frac{t}{1 - e^{-t}} \right) \]

Polychronakos ’13; JT ’15, JT ’17
Such $F$ introduces a (not too strong) interaction among the eigenvalues. For some values of $r, g$ an asymmetric configuration can become stable.

It corresponds to the "standard" symmetry broken phase.

\[ \rho(x) \]

\[ \rho(x) \]

\[ \rho(x) \]
Hermitian matrix model
A very good qualitative agreement. A very good quantitative agreement in the critical coupling.

Different value for the critical mass parameter and different behaviour of the asymmetric transition line for large $-r$.

We need to include $R$ in a nonperturbative way.

work in progress with M. Šubjaková
Recall the perturbative action

\[ S_{\text{eff}}(M) = \frac{1}{2} \left[ \varepsilon \frac{1}{2} (c_2 - c_1^2) - \varepsilon^2 \frac{1}{24} (c_2 - c_1^2)^2 + \varepsilon^4 \frac{1}{2880} (c_2 - c_1^2)^4 \right] - \]

\[ - \varepsilon^4 \frac{1}{3456} \left[ (c_4 - 4c_3c_1 + 6c_2c_1^2 - 3c_1^4) - 2 (c_2 - c_1^2)^2 \right]^2 - \]

\[ - \varepsilon^3 \frac{1}{432} \left[ c_3 - 3c_1c_2 + 2c_1^3 \right]^2 \approx \]

\[ \approx \frac{1}{2} F_2[t_2] + F_3[t_3] + F_4[t_4 - 2t_2^2] \]
Find a function which gives a correct perturbative expansion and behaves well close to the triple point. E.g.

\[ n \log \left( 1 + A \frac{t^2}{n} \right), \frac{1}{\left(1 + A \frac{t^2}{n}\right)^2 - 1}, \]

\[ -An \log \left( 1 + \frac{t^2}{n} \right), A \left( \frac{1}{\left(1 + A \frac{t^2}{n}\right)^2 - 1} \right). \]

So far it either does barely anything or completely ruins the model.
Symmetry breaking in noncommutative field theory is (very) different than in the commutative case.

Matrix models are a great tool to analyze the(se) properties of scalar field theories on fuzzy spaces, and beyond.
Outlook

To do list.

- Find (a more) complete understanding of the matrix model.
- Investigate matrix models corresponding to spaces beyond the fuzzy sphere.
- Investigate matrix models corresponding to theories without the UV/IR mixing.
Thank you for your attention!
If time permits I

- Recall the perturbative action

\[ S_{eff}(M) = \frac{1}{2} \left[ \varepsilon \frac{1}{2} (c_2 - c_1^2) - \varepsilon^2 \frac{1}{24} (c_2 - c_1^2)^2 + \varepsilon^4 \frac{1}{2880} (c_2 - c_1^2)^4 \right] - \]

\[ - \varepsilon^4 \frac{1}{3456} \left[ (c_4 - 4c_3c_1 + 6c_2c_1^2) - 3c_1^4 - 2(c_2 - c_1^2)^2 \right]^2 - \]

\[ - \varepsilon^3 \frac{1}{432} \left[ c_3 - 3c_1c_2 + 2c_1^3 \right]^2 \]

\[ = \frac{1}{2} - \frac{1}{2} c_2 - \frac{1}{4} c_1^2 - \frac{1}{24} c_2^2 - \frac{1}{432} c_3^2 - \frac{1}{3456} c_4^2 + \ldots \]

- This part can be interpreted as an additional two-particle interaction.
Recall the perturbative action

\[ S_{\text{eff}}(M) = \frac{1}{2} c_2 - \frac{1}{4} c_1^2 - \frac{1}{24} c_2^2 - \frac{1}{432} c_3^2 - \frac{1}{3456} c_4^2 + \ldots \]

Function of the form

\[ S_{\text{eff}} = \sum_{i,j} a \log(1 - b \lambda_i \lambda_j) \]

with \( a = 3/2, b = 1/6 \) correctly reproduces all four known coefficients.
If time permits II

Investigate matrix models corresponding to theories without the UV/IR mixing.

- For a noncommutative theory with no UV/IR mixing, the extra phase should not be present in the commutative limit of the phase diagram.
  - B.P. Dolan, D. O’Connor and P. Prešnajder [arXiv:0109084],
  - H. Grosse and R. Wulkenhaar [arXiv:0401128],
- Understanding the phase diagram of such theories, especially mechanism of the removal of the striped phase could teach us a lot technically and conceptually.
If time permits III